

# On the origin of inflation

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In this paper we discuss a space-time having the topology of  $S^3 \times \mathbb{R}$  but with different smoothness structure. This space-time is not a global hyperbolic space-time. Especially we obtain a time line with a topology change of the space from the 3-sphere to a homology 3-sphere and back but without a topology-change of the space-time. Among the infinite possible smoothness structures of this space-time, we choose a homology 3-sphere with hyperbolic geometry admitting a homogenous metric. Then the topology change can be described by a time-dependent curvature parameter  $k$  changing from  $k = +1$  to  $k = -1$  and back. The solution of the Friedman equation for dust matter ( $p = 0$ ) after inserting this function shows an exponential growing which is typical for inflation. In contrast to other inflation models, this process stops after a finite time.

Because of the influx of observational data, recent years have witnessed enormous advances in our understanding of the early universe. To interpret the present data, it is sufficient to work in a regime in which space-time can be taken to be a smooth continuum as in general relativity, setting aside fundamental questions involving the deep Planck regime. However, for a complete conceptual understanding as well as interpretation of the future, more refined data, these long-standing issues will have to be faced squarely. As an example, can one show from first principles that the smooth space-time of general relativity is valid at the onset of inflation? In this paper we will focus mainly on this question about the origin of inflation. Inflation is today the main theoretical framework that describes the early Universe and that can account for the present observational data [16]. In thirty years of existence [14, 18], inflation has survived, in contrast with earlier competitors, the tremendous improvement of cosmological data. In particular, the fluctuations of the Cosmic Microwave Background (CMB) had not yet been measured when inflation was invented, whereas they give us today a remarkable picture of the cosmological perturbations in the early Universe. In nearly all known models, the inflation period is caused by one or more scalar field(s) [17]. A different approach is Loop quantum cosmology containing also the inflation scenario in special cases [3, 4].

In this paper we try to derive an inflationary phase from first principles. The analysis of the WMAP data seems to imply that our universe is a compact 3-manifold with a slightly positive curvature [26]. Therefore we choose the topology of the space-time to be  $S^3 \times \mathbb{R}$ . Clearly this space-time admits a Lorentz metric (given by the topological condition to admit a non-vanishing vector field normal to  $S^3$ ). But we weaken the condition of global hyperbolicity otherwise it induces a diffeomorphism [6, 7] to  $S^3 \times \mathbb{R}$ . The space-time has the topology of  $S^3 \times \mathbb{R}$  and is smoothable (i.e. a smooth 4-manifold) [23] but (we assume) is not diffeomorphic to  $S^3 \times \mathbb{R}$ . What does it mean? Every manifold is defined by a collection of charts, the atlas, including also the transition functions between the charts. From the physical point of view, charts are the reference frames. The transition functions define the structure of the manifold, i.e. transition functions are homeomorphisms (topological manifold) or diffeomorphisms (smooth manifold). Two (smooth) atlases are compatible (or equivalent) if their union is a (smooth) atlas again. The equivalence class (the maximal atlas) is called a differential structure [28]. In dimension smaller than 4, there is only one differential structure, i.e. the topology of these manifolds define uniquely its smoothness properties. In contrast, beginning with dimension 4 there is the possibility of more than one differential structure. But 4-manifolds are really special: here there are many examples of 4-manifolds with infinite many differential structures (countable for compact and uncountable for non-compact 4-manifolds including  $\mathbb{R}^4$ ). Among these differential structures there is one exceptional, the standard differential structure. We will illustrate these standard structure for our space-time  $S^3 \times \mathbb{R}$ . The 3-sphere  $S^3$  has an unique differential structure (the standard differential structure) which extends to  $S^3 \times \mathbb{R}$ . All other differential structures (also called misleadingly "exotic smoothness structures") can never split smoothly into  $S^3 \times \mathbb{R}$ . We denote it by  $S^3 \times_{\theta} \mathbb{R}$ . Our main hypothesis is now:

**Main hypothesis:** *The space-time has the topology  $S^3 \times \mathbb{R}$  but the differential structure  $S^3 \times_{\theta} \mathbb{R}$ .*

In [11] the first  $S^3 \times_{\theta} \mathbb{R}$  was constructed and we will use this construction here. One starts with a homology 3-sphere  $P$ , i.e. a compact 3-manifold  $P$  with the same homology as the 3-sphere but non-trivial fundamental group, see [8]. The Poincare sphere is one example of a homology 3-manifold. Now we consider the 4-manifold  $P \times [0, 1]$  with the

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same fundamental group  $\pi_1(P \times [0, 1]) = \pi_1(P)$ . By a special procedure (the plus construction, see [19, 25]), one can "kill" the fundamental group  $\pi_1(P)$  in the interior of  $P \times [0, 1]$ . This procedure will result in a 4-manifold  $W$  with boundary  $\partial W = -P \sqcup S^3$  ( $-P$  is  $P$  with opposite orientation), a so-called cobordism between  $P$  and  $S^3$ . The gluing  $-W \cup_P W$  along  $P$  with the boundary  $\partial(-W \cup_P W) = -S^3 \sqcup S^3$  defines one piece of the exotic  $S^3 \times_{\Theta} \mathbb{R}$ . The whole construction can be extended to both directions to get the desired exotic  $S^3 \times_{\Theta} \mathbb{R}$  (see [11, 15] for the details). There is one critical point in the construction: the 4-manifold  $W$  is not a smooth manifold. As Freedman [12] showed, the 4-manifold  $W$  always exists topologically but, by a result of Gompf [13] (using Donaldson [9]), not smoothly (i.e. it does not exist as a smooth 4-manifold). The 4-manifold  $-W \cup_P W$  is also non-smoothable and we will get a smoothness structure only for the whole non-compact  $S^3 \times_{\Theta} \mathbb{R}$  (see [23]). But  $S^3 \times_{\Theta} \mathbb{R}$  contains  $-W \cup_P W$  with the smooth cross section  $P$ . From the physical point of view we interpret  $-W \cup_P W$  as a time line of a cosmos starting as 3-sphere changing to the homology 3-sphere  $P$  and changing back to the 3-sphere. But this process is part of every exotic smoothness structure  $S^3 \times_{\Theta} \mathbb{R}$ , i.e. we obtain the mathematical fact

**Fact:** *In the space-time  $S^3 \times_{\Theta} \mathbb{R}$  we have a change of the spatial topology from the 3-sphere to some homology 3-sphere  $P$  and back but without changing the topology of the space-time.*

Now we have to discuss the choice of the homology 3-sphere  $P$ . At first, usually every homology 3-sphere is the boundary of a topological, contractable 4-manifold [12] but this homology 3-sphere  $P$  never bounds a **smooth**, contractable 4-manifold. Secondly, every homology 3-sphere can be constructed by using a knot [24]. One starts with the complement  $S^3 \setminus (D^2 \times K)$  of a knot  $K$  (a smooth embedding  $S^1 \rightarrow S^3$ ) and glue in a solid torus  $D^2 \times S^1$  using a special map (a  $\pm 1$  Dehn twist). The resulting 3-manifold  $\Sigma(K)$  is a homology 3-sphere. For instance the trefoil knot  $3_1$  (in Rolfsen notation [24]) generates the Poincare sphere by this method (with  $-1$  Dehn twist).

Our model  $S^3 \times_{\Theta} \mathbb{R}$  starts with a 3-sphere as spatial topology. Now we will be using a powerful tool for the following argumentation, Thurston's geometrization conjecture [27] proved by Perelman [20–22]. According to this theory, only the 3-sphere and the Poincare sphere carry a homogenous metric of constant positive scalar curvature (spherical geometry or Bianchi IX model) among all homology 3-spheres. Also other homology 3-spheres [29] are able to admit a homogenous metrics. There is a close relation between Thurston's geometrization theory and Bianchi models in cosmology [1, 2]. Most of the (irreducible) homology 3-spheres have a hyperbolic geometry (Bianchi V model), i.e. a homogenous metric of negative curvature. Here we will choose such a hyperbolic homology 3-sphere. As an example we choose the knot  $8_{10}$  leading to the hyperbolic homology 3-sphere  $\Sigma(8_{10})$ . So we will give an overview of our assumptions:

1. The space-time is  $S^3 \times_{\Theta} \mathbb{R}$  (with topology  $S^3 \times \mathbb{R}$ ) containing a homology 3-sphere  $P$  (as cross section).
2. This homology 3-sphere  $P$  is a hyperbolic 3-manifold (with negative scalar curvature).

According to the mathematical fact above, the 3-sphere is changed to  $P$  and back in  $S^3 \times_{\Theta} \mathbb{R}$ , i.e. we obtain a topology change of the spatial cosmos. Now we will study the geometry and topology changing process more carefully. Let us consider the Robertson-Walker metric (with  $c = 1$ )

$$ds^2 = dt^2 - a(t)^2 h_{ik} dx^i dx^k$$

with the scaling function  $a(t)$ . At first we assume a space-time  $S^3 \times \mathbb{R}$  with increasing function  $a(t)$  fulfilling the Friedman equations

$$\left( \frac{\dot{a}(t)}{a(t)} \right)^2 + \frac{k}{a(t)^2} = \kappa \frac{\rho}{3} \quad (1)$$

$$2 \left( \frac{\ddot{a}(t)}{a(t)} \right) + \left( \frac{\dot{a}(t)}{a(t)} \right)^2 + \frac{k}{a(t)^2} = -\kappa p \quad (2)$$

derived from Einsteins equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad (3)$$

with the gravitational constant  $\kappa$  and the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu\nu} \quad (4)$$

with the (time-dependent) energy density  $\rho$  and the (time-dependent) pressure  $p$ . The spatial cosmos has the scalar curvature  ${}^3R$

$${}^3R = \frac{k}{a^2}$$

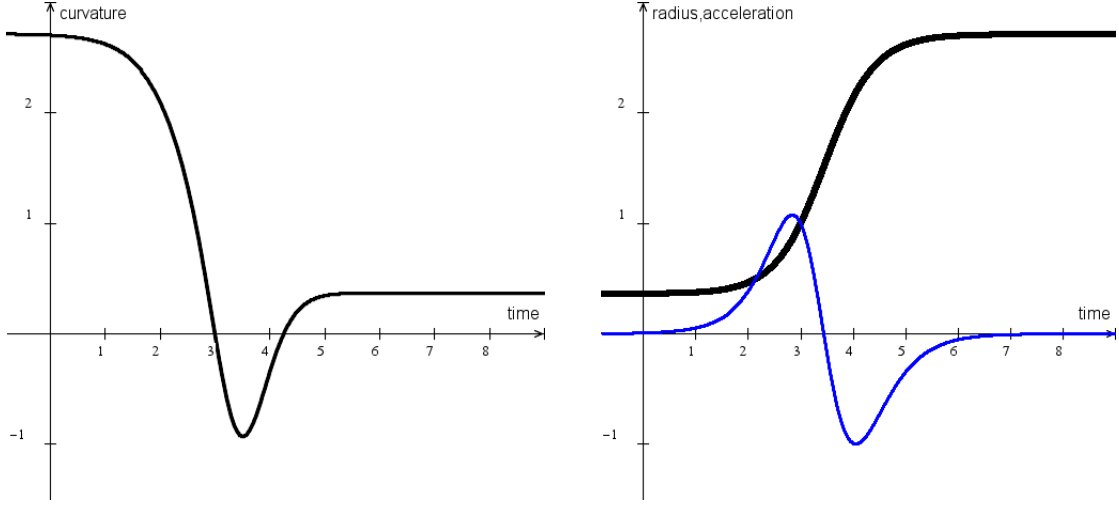


Figure 1: time-dependent curvature  $k(t)$  with  $t_0 = 3, \zeta = 1, \rho_0 = 1$  (left fig.) and the solution  $a(t)$  (thick curve) as well the acceleration  $\ddot{a}(t)$  (right fig.)

from the 3-metric  $h_{ik}$  and we obtain the 4-dimensional scalar curvature  $R$

$$R = \frac{6}{a^2} (\ddot{a} \cdot a + \dot{a}^2 + k) . \quad (5)$$

Let us consider the model  $S^3 \times \mathbb{R}$  with positive spatial curvature  $k = +1$ . In case of dust matter ( $p = 0$ ) only, one obtains a closed universe. Now we consider our model of an exotic  $S^3 \times_{\Theta} \mathbb{R}$ . As explained above, the foliation of  $S^3 \times_{\Theta} \mathbb{R}$  must contain a hyperbolic homology 3-sphere  $P = \Sigma(8_{10})$  (with negative scalar curvature). But then we have a transition from a space with positive curvature to a space with negative curvature and back. To model this behavior, we consider a time-dependent parameter  $k(t)$  in the curvature

$${}^3R(t) = \frac{k(t)}{a^2}$$

with the following conditions:

1. The change of the geometry from spherical  $k > 0$  to hyperbolic  $k < 0$  happens at  $t_0$ ,
2.  $k(t) > 0$  for  $t \ll t_0$  and  $t \gg t_0$ .

Furthermore,  $k(t)$  for  $t \ll t_0$  must be larger than  $k(t)$  for  $t \gg t_0$ . The change of the topology is an abrupt process which can be modeled by a tanh function. Putting all these conditions together we choose

$$k(t) = \rho_0 \exp(-\tanh(\zeta(t - t_0))) - \zeta^2 (1 - \tanh^2(\zeta(t - t_0)))^2 \exp(2 \cdot \tanh(\zeta(t - t_0))) .$$

This function is plotted in the left figure of Fig. 1 for special parameters confirming the conditions above. Now we consider the special Friedman equation

$$\left( \frac{\dot{a}(t)}{a(t)} \right)^2 + \frac{k(t)}{a(t)^2} = \frac{\rho_0}{a(t)^3}$$

with dust matter ( $p = 0$ ) (having the scaling behavior  $\rho \sim a^{-3}$ ) by inserting the time-dependent  $k(t)$  above. The solution  $a(t)$  of this equation is given by

$$a(t) = \exp(\tanh(\zeta(t - t_0)))$$

also visualized in the right figure of Fig. 1. Especially we obtain a *positive acceleration*  $\ddot{a}(t)$  in some time interval. The *growing rate is exponential* as required by inflation. Furthermore the acceleration  $\ddot{a}(t)$  is also negative or the *inflation process stops*. Finally the topology change of the space

$$\text{spherical 3-sphere} \longrightarrow \text{hyperbolic homology 3-sphere} \longrightarrow \text{spherical 3-sphere}$$

produces an exponential growing rate of  $a(t)$ , also called inflation. But in contrast to the usual inflation models, we derive this behavior from first principles using the space-time  $S^3 \times_{\theta} \mathbb{R}$  (with a non-standard differential structure). Furthermore *in our inflation model, the exponential growing stops, i.e. our inflation is not eternal.*

But what is about the inflation without quantum effects? Fortunately, there is growing evidence that the differential structures constructed above (i.e. exotic smoothness in dimension 4) is directly related to quantum gravitational effects [5, 10]. We will further investigate this interesting direction in our future work.

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